# Infinite Series and their Convergence

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(i) Defⁿ- **Series:-** A Series is a Succession of quantities which are formed in order according to some definite law.

The successive terms of a Series will be denoted by $u_1, u_2, u_3$ - - - -.

(ii) **Finite Series:-** If a Series terminates after a certain number of terms, it is said to be in finite Series.

(iii) **Infinite Series:-** If there be an endless succession of terms, the Series is said to be infinite.

An infinite Series is denoted by

$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$

or, $\sum_{n=1}^{\infty} u_n$ or by $\sum u_n$.

(iv) **Convergent Series:-** If the sum of the first $n$ terms of a Series tends to a finite limit $S$, so that the sum can be sufficiently increasing $n$, be made to differ from $S$ by less than any assigned quantity however small, the Series is said to be Convergent and $S$ is called is Sum.

Thus, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \cdots$ is a Convergent Series whose Sum is 2.

(V) Divergent Series:- If the Sum of the first n terms of a Series increases numerically without limit as n is increased indefinitely, the Series is said to be divergent.

Thus, $1+2+3+4 \cdots$ is a divergent Series.

QNo- State and Prove Comparison Test.

Statement of Comparison Test:-

If $\sum a_n$ & $\sum b_n$ be two Series of positive term Such that,

$$\frac{a_n}{a_{n+1}} > \frac{b_n}{b_{n+1}}, \text{ for all } n \geq m.$$

Then, (i) If $\sum b_n$ Converges, $\sum a_n$ also Converges.

(ii) If $\sum a_n$ diverges, $\sum b_n$ also diverges.

Proof:- (i) from Statement (i), we have

$$\frac{a_m}{a_{m+1}} > \frac{b_m}{b_{m+1}}$$

Replacing m by $m+1, m+2, \cdots, n-1$.

$$\frac{a_{m+1}}{a_{m+2}} > \frac{b_{m+1}}{b_{m+2}},$$

$$\frac{a_{m+2}}{a_{m+3}} > \frac{b_{m+2}}{b_{m+3}}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\frac{a_{n-1}}{a_n} > \frac{b_{n-1}}{b_n}$$

Now, multiplying the Corresponding Sides of the above inequalities, we have

$$\frac{a_{m+1}}{a_n} > \frac{b_{m+1}}{b_n} \text{ for all } n \geq m.$$

i.e. $a_n < \left(\frac{a_{m+1}}{b_{m+1}}\right) b_n$ —— (1)

$\because \frac{a_{m+1}}{b_{m+1}} = +ve \text{ nos} = K$

$a_n < K b_n$ for all $K \geq n$.

Hence the result follows by Comparison Test.

QNo- Prove that the infinite Series,

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \text{ to } \infty$$

is Convergent if $p>1$ and divergent if $p \leq 1$.

Proof- As the given series is a Series of Positive terms, so its Convergence is not affected by grouping the terms in brackets in any way we wish.

Case (i) when $p>1$

let $S = \frac{1}{p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

Here, we group the terms as follows,

$$1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots \quad (A)$$

$$\because \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p}$$

and $\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$ i.e. $\frac{4}{4^p}$

on adding, we have

$$S < 1 + \frac{2}{2^p} + \frac{4}{4^p} + \text{---} \text{ to } \infty$$

It is in G.P. whose Ist term is $a = 1$ & c.r. $r = \frac{2}{2^p}$

$$S < \frac{1}{1 - \frac{2}{2^p}}$$

Hence, the Series S is Convergent.

<u>Case (ii)</u> when, $p = 1$

Then the Series becomes,

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \text{------} \text{ to } \infty.$$

Now, grouping it like this,

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \text{----} \text{ to } \infty.$$

$$\because 1 + \frac{1}{2} = 1 + \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} \text{ i.e. } \frac{2}{4} \text{ i.e. } \frac{1}{2}$$

Now, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ i.e. $\frac{4}{8}$ i.e. $\frac{1}{2}$

and So on.

on adding, we have

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$$S > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \text{---} \text{ to } \infty$$

which is divergent Series.

Hence, S is divergent.

<u>Case (iii)</u>:- when $p < 1$

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} > \frac{1}{2}$$

$$\frac{1}{3^p} > \frac{1}{3}$$

$$\frac{1}{4^p} > \frac{1}{4}$$

on adding, $S > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ to $\infty$.

which is divergent series.

Hence, S is divergent when $p < 1$.

$\therefore$ the series will be converges when $p > 1$ and divergent when, $p \leq 1$.